A generalization of Kummer theory to Hopf-Galois extensions

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From this new perspective, we can study many extensions for which $\zeta_n \notin K$ and translate results as the one above to this new setting.

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The conjugates of α are $\alpha, \zeta_n\alpha, \ldots, \zeta_n^{n-1}\alpha$.

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The element α is an eigenvector of all the automorphisms in G. We call it a G-**eigenvector**.

Proposition

Let L/K be a Galois extension with group G. Then L/K admits some primitive element which is a G-eigenvector if and only if $\zeta_n \in K$ and L/K is cyclic.

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This characterization can be extended to arbitrary Kummer extensions.

Idea: Any Kummer extension can be described as a compositum of cyclic extensions and the eigenvectors property *lifts* in compositums.

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Theorem

Let L/K be a Galois extension of K with group G. Then, L/K admits some finite generating set of G-eigenvectors if and only if $\zeta_n \in K$ and L/K is Kummer.

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If $L = K(\sqrt[n]{a_1}, \dots, \sqrt[n]{a_k})$ is such an extension, a finite generating set of G-eigenvectors is $\{\sqrt[n]{a_1}, \dots, \sqrt[n]{a_k}\}$.

Definition

Let L/K be an H-Galois extension. We say that $\alpha \in L$ is an H-eigenvector if for every $h \in H$ there is some $\lambda(h) \in K$ such that

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It is well known that adjoining $\omega = \sqrt[3]{2}$ to $\mathbb{Q}$ does not give a Galois extension of $\mathbb{Q}$, i.e., there is no set of automorphisms of $\mathbb{Q}(\omega)$ whose set of common fixed elements is precisely $\mathbb{Q}$. However, one can define the following linear maps s, c from $\mathbb{Q}(\omega)$ to itself by

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Therefore, ω is an H-eigenvector and $L/\mathbb{Q}$ is H-Kummer.

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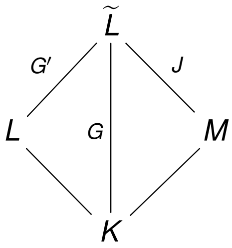

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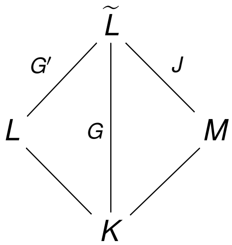

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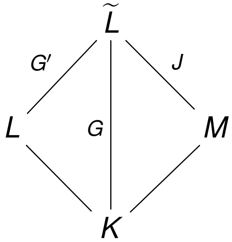

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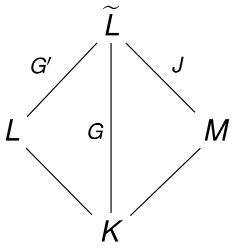

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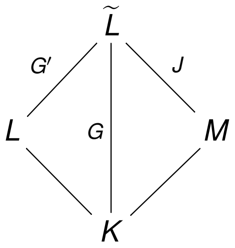

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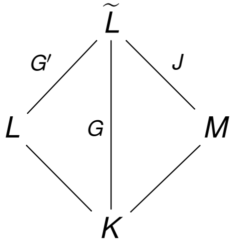

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Identifying $J \cong \lambda(J)$, the underlying Hopf algebra is $H = M[J]^{G'}$.

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where $\lambda(h) = \sum_i m_i(h)\lambda_i \in M$. But $\lambda(h) = \frac{h\cdot\alpha}{\alpha} \in L \cap M = K$.

Theorem

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- $L \cap K(\zeta_n) = K$ *and* L/K *is simple radical.*
- L/K *is an a.c.G. extension with complement* $K(\zeta_n)$ *and admits some primitive element which is an* H*-eigenvector, where* H *is the a.c.G. structure on* L/K *corresponding to* $K(\zeta_n)$.

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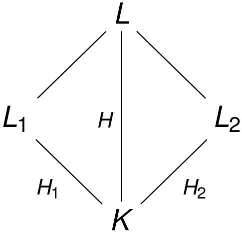

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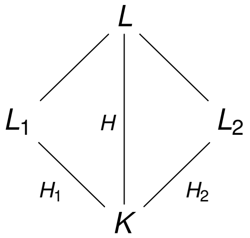


Let H_i be the a.c.G. structure corresponding to $K(\zeta_{n_i})$, $i \in \{1, 2\}$.

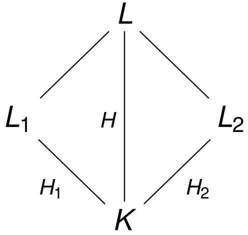
Let $L = K(\sqrt[n]{a_1}, \dots, \sqrt[n]{a_k})$ with $L \cap K(\zeta_n) = K$.

L/K is a.c.G. with complement $M = K(\zeta_n)$ and we can consider the a.c.G. structure H corresponding to $K(\zeta_n)$.

Natural question: Is L/K H-Kummer?

Natural strategy: Work with the simple radical extensions $L_i = K(\sqrt[n]{a_i})$ and *lift* the information to their compositums.

Set $k = 2$, so that $L = L_1 L_2$, $L_i = K(\sqrt[n]{a_i})$.



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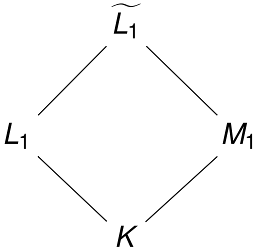

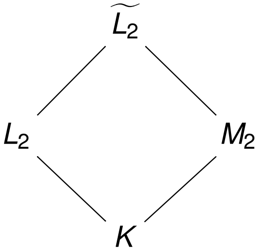

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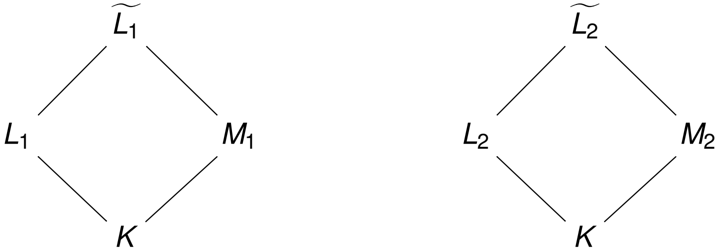

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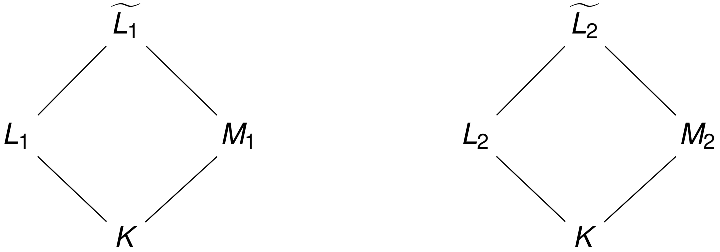

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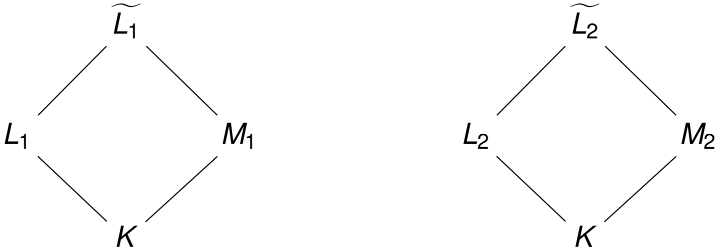

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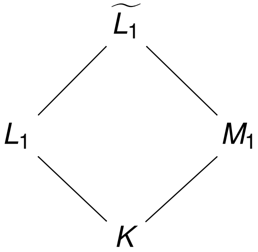

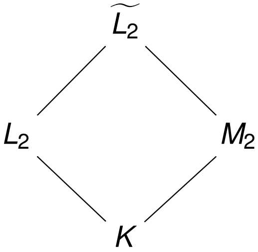

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This is the condition that the epimorphism

$$L_1 \otimes_K L_2 \longrightarrow L_1L_2$$

is an isomorphism.

Definition

Two linearly disjoint a.c.G. extensions L_1/K, L_2/K with complements M_1, M_2 are said to be ***strongly disjoint*** *if*

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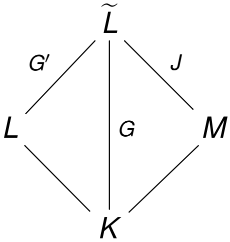
Since $M_1, M_2/K$ are Galois, strong disjointness gives $L_i \otimes_K M_j \hookrightarrow \widetilde{L}$.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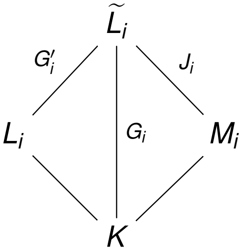
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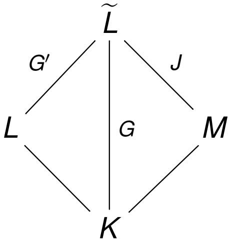

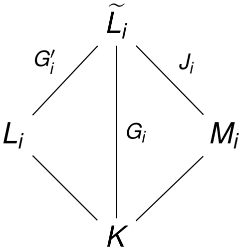

There is an embedding $G \hookrightarrow G_1 \times G_2$, by which

$$G' \hookrightarrow G'_1 \times G'_2, \quad J \hookrightarrow J_1 \times J_2.$$

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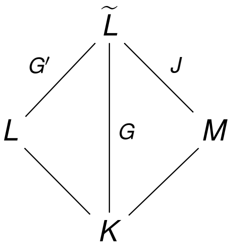

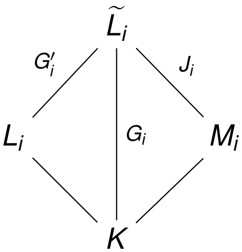

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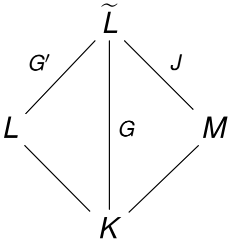

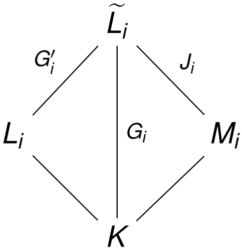

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There is a bijection

$$\begin{array}{rccc} \psi \colon & G/G' & \longrightarrow & G_1/G_1' \times G_2/G_2' \\ & g_1 g_2 G' & \longmapsto & (g_1 G_1', g_2 G_2') \end{array}$$

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Then there is a well defined map

$$\iota\colon \mathrm{Perm}(G_1/G_1') \times \mathrm{Perm}(G_2/G_2') \longrightarrow \mathrm{Perm}(G/G').$$

It is easy to check that ι is a monomorphism.

Proposition

If N_i is a regular subgroup of $\mathrm{Perm}(G_i/G_i')$ normalized by $\lambda_i(G_i)$ for every $i \in \{1,2\}$, then $N = \iota(N_1 \times N_2)$ is a regular subgroup of $\mathrm{Perm}(G/G')$ normalized by $\lambda(G)$.

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Definition

Let $H_i = \widetilde{L_i}[N_i]^{G_i}$ be a Hopf-Galois structure on L_i/K, $i \in \{1,2\}$. The Hopf-Galois structure $H = \widetilde{L}[N]^G$ on L/K will be called the ***product Hopf-Galois structure*** *of H_1 and H_2.*

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Proposition

Let H be the product Hopf-Galois structure of H_1 and H_2. Then:

1. $H \cong H_1 \otimes_K H_2$ *as* K*-algebras.*
2. *If* $h_i \in H_i$ *and* $\alpha_i \in L_i$ *for* $i \in \{1, 2\}$, *then*

$$(h_1 h_2) \cdot (\alpha_1 \alpha_2) = (h_1 \cdot \alpha_1)(h_2 \cdot \alpha_2).$$

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Let $\rho_{H_i} \colon H_i \longrightarrow \mathrm{End}_K(L_i)$ and $\rho_H \colon H \longrightarrow \mathrm{End}_K(L)$ be the actions of H_i on L_i for $i \in \{1, 2\}$ and of H on L, respectively. Then

$$\rho_H = \rho_{H_1} \otimes_K \rho_{H_2}.$$

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Let M_i be the complement of L_i/K and let $M = M_1M_2$. If H_i is the a.c.G. structure on L_i/K corresponding to M_i, then H is the a.c.G. structure on L/K corresponding to M.

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Theorem

Let L/K be a strongly decomposable H-Galois extension. The following statements are equivalent:

- $L \cap K(\zeta_n) = K$ *and* L/K *is radical.*
- L/K *is an a.c.G. extension with complement* $K(\zeta_n)$ *and* L/K *is* H*-Kummer, where* H *is the a.c.G. extension on* L/K *corresponding to* $K(\zeta_n)$*.*

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If $L = K(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_k})$ is such an extension, a finite generating set of H-eigenvectors is $\{\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_k}\}$.

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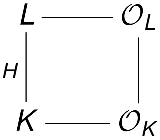

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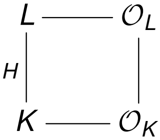

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Let $L = \mathbb{Q}(\omega)$, $\omega = \sqrt[3]{2}$. We have seen that $\{1, \omega, \omega^2\}$ are H-eigenvectors of L.

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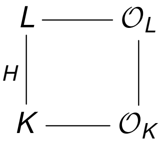

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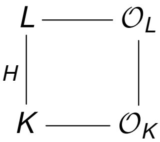

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Let $L = K(\sqrt[n]{a})$ be a simple radical extension such that $L \cap K(\zeta_n) = K$ and $\mathcal{O}_L = \mathcal{O}_K[\sqrt[n]{a}]$. Then $\mathcal{O}_L$ is $\mathfrak{A}_H$-free, where H is the a.c.G. structure corresponding to $K(\zeta_n)$.

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Corollary

Let $L = \mathbb{Q}(\sqrt[n]{a})$ be such that $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ and $a^p \not\equiv a \pmod{p^2}$ for every prime divisor p of n. Then $\mathcal{O}_L$ is $\mathfrak{A}_H$-free.

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Proposition

Let $L = K(\sqrt[n_1]{a_1}, \dots, \sqrt[n_k]{a_k})$ and call $L_i = K(\sqrt[n_i]{a_i})$. Assume:

- *$L_i \cap K(\zeta_{n_i}) = K$ for every $1 \leq i \leq k$.*
- *L_i/K and L_j/K are strongly and arithmetically disjoint.*
- *$\mathcal{O}_{L_i} = \mathcal{O}_K[\sqrt[n_i]{a_i}]$ for every $1 \leq i \leq k$.*

Then $\mathcal{O}_L$ is $\mathfrak{A}_H$-free.

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Thank you for your attention